

F -SIGNATURE EXISTS

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ABSTRACT. Suppose R is a d -dimensional reduced F -finite Noetherian local ring with prime characteristic $p > 0$ and perfect residue field. Let R^{1/p^e} be the ring of p^e -th roots of elements of R for $e \in \mathbb{N}$, and let a_e denote the maximal rank of a free R -module appearing in a direct sum decomposition of R^{1/p^e} . We show the existence of the limit $s(R) := \lim_{e \rightarrow \infty} \frac{a_e}{p^e d}$, called the F -signature of R . This invariant – which can be extended to all local rings in prime characteristic – was first formally defined by C. Huneke and G. Leuschke [HL02] and has previously been shown to exist only in special cases. The proof of our main result is based on the development of certain uniform Hilbert-Kunz estimates of independent interest. Additionally, we analyze the behavior of the F -signature under finite ring extensions and recover explicit formulae for the F -signatures of arbitrary finite quotient singularities

1. INTRODUCTION

Every ring R with prime characteristic $p > 0$ comes endowed with a Frobenius or p -th power endomorphism. The existence of an R -module section of Frobenius, called an F -splitting, has strong algebraic and geometric consequences. Historically, F -splittings have featured prominently throughout diverse fields of mathematics, and applications of these techniques include the well-known theorems of M. Hochster and J. L. Roberts [HR74] together with numerous results in representation theory [BK05]. In this paper, we answer an important question which has remained open for over a decade by showing the existence of a local numerical invariant – the F -signature – which roughly characterizes the asymptotic growth of the number of splittings of the iterates of Frobenius.

2000 *Mathematics Subject Classification.* 14B05, 13A35.

This material is based upon work supported by the National Science Foundation under Award No. 1004344.

More precisely, let R be complete d -dimensional reduced Noetherian local ring with prime characteristic $p > 0$ and perfect residue field $k = k^p$. For $e \in \mathbb{N}$ the inclusion $R \subseteq R^{1/p^e}$ into the corresponding ring of p^e -th roots of elements of R is naturally identified with the e -th iterate of the Frobenius endomorphism. Let a_e denote the largest rank of a free R -module appearing in a direct sum decomposition of R^{1/p^e} . In other words, we may write $R^{1/p^e} = R^{\oplus a_e} \oplus M_e$ as R -modules where M_e has no free direct summands. The number a_e is called the e -th *Frobenius splitting number* of R , and collectively these numbers encode subtle information about the action of the Frobenius endomorphism on R .

The primary goal of this paper is to analyze the asymptotic behavior of the sequence $\{a_e\}_{e \in \mathbb{N}}$ by showing the existence of the limit $s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$, called the *F-signature* of R .

Main Result (Theorem 4.9). *The F-signature $s(R) := \lim_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$ exists.*

This invariant was first formally defined by C. Huneke and G. Leuschke [HL02] and captures delicate information about the singularities of R . For example, the *F-signature* of the two-dimensional rational double-points¹ (A_n) , (D_n) , (E_6) , (E_7) , (E_8) is the reciprocal of the order of the group defining the corresponding quotient singularity [HL02, Example 18]. We recover herein similar formulae for the *F-signatures* of arbitrary finite quotient singularities (Corollary 4.13; *c.f.* [Yao06, Remark 4.7]).

It is quite natural to expect the *F-signature* to measure the singularities of R . Indeed, when R is regular, R^{1/p^e} itself is a free R -module of rank p^{ed} . Thus, for general R , the *F-signature* asymptotically compares the number of direct summands of R^{1/p^e} isomorphic to R with the number of such summands one would expect from a regular local ring of the same dimension. We will see (*c.f.* Theorem 4.16) that $s(R) \leq 1$ with equality if and only if R is regular; furthermore, assuming its existence, it has been shown by I. Aberbach and G. Leuschke [AL03] that the *F-signature* is positive if and only if R is strongly *F-regular*.

In light of lingering doubts concerning existence, previous research on the *F-signature* has largely been done through the use of so-called lower and upper *F-signatures*. These are given by $s^-(R) := \liminf_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$ and $s^+(R) := \limsup_{e \rightarrow \infty} \frac{a_e}{p^{ed}}$, respectively. Thus, the goal of this paper is simply to show the equality $s^-(R) = s^+(R)$ of the lower and upper *F-signatures* in full generality.

Historically, the *F-signature* first appeared implicitly in the work of K. Smith and M. Van den Bergh [SVdB97]. However, since the beginning of its formal study in [HL02], the existence of the *F-signature* limit has been shown only in special cases. These include Gorenstein local rings [HL02], local rings that are Gorenstein on the punctured spectrum [Yao06], affine semigroup rings [Sin05], general \mathbb{N} -graded rings [AE06], and local rings that are \mathbb{Q} -Gorenstein on the punctured spectrum [AE06]. Most recently, I. Aberbach [Abe08] uses certain degree bounds on local cohomology modules² to treat the case of rings whose

¹Here it is necessary to assume that $p \geq 7$ to avoid pathologies in low characteristic.

²These bounds are known to hold for rings which are essentially of finite type over a field [Vra00].

non- \mathbb{Q} -Gorenstein locus has dimension at most one. It should be noted that the proof given herein will use rather elementary techniques in comparison.

Let us sketch the proof of the existence of the F -signature. Recall that, according to the famous result of P. Monsky [Mon83], for any \mathfrak{m} -primary ideal $I = \langle x_1, \dots, x_r \rangle$ we may define the *Hilbert-Kunz multiplicity of I along R*

$$e_{HK}(I; R) := \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell(R/I^{[p^e]})$$

where $\ell(_)$ denotes the length function on Artinian R -modules and $I^{[p^e]} := \langle x_1^{p^e}, \dots, x_r^{p^e} \rangle$. In Section 3, we give a variant on the original proof of the existence of Hilbert-Kunz multiplicity which carefully tracks certain uniform estimates. The most important of these is the following (Proposition 3.4): for any d -dimensional reduced F -finite ring (R, \mathfrak{m}, k) , there exists a positive constant C such that for all $e, e' \in \mathbb{N}$ and every ideal I of R containing $\mathfrak{m}^{[p^e]}$ we have

$$\left| \ell(R/I) - \frac{1}{p^{e'd}} \ell(R/I^{[p^{e'}]}) \right| \leq Cp^{e(d-1)} .$$

In Section 4, building on the works of Y. Yao [Yao06] as well as F. Enescu and I. Aberbach [AE05], for each fixed $e \in \mathbb{N}$ we consider the ideal

$$I_e = \{r \in R \mid \phi(r^{1/p^e}) \in \mathfrak{m} \text{ for all } \phi \in \text{Hom}_R(R^{1/p^e}, R)\} .$$

It is easy to see that $\mathfrak{m}^{[p^e]} \subseteq I_e$, so it follows (Corollary 3.7) by the uniform estimate above that

$$\lim_{e \rightarrow \infty} \left(\frac{1}{p^{ed}} (\ell(R/I_e) - e_{HK}(I_e; R)) \right) = 0 .$$

Since one can show $a_e = \ell(R/I_e)$ for all $e \in \mathbb{N}$, to prove the existence of the F -signature it suffices to show the sequence $\{\frac{1}{p^{ed}} e_{HK}(I_e; R)\}_{e \in \mathbb{N}}$ approaches a limit. This follows by noting that $I_e^{[p]} \subseteq I_{e+1}$ for all $e \in \mathbb{N}$, and thus $\{\frac{1}{p^{ed}} e_{HK}(I_e; R)\}_{e \in \mathbb{N}}$ is non-increasing (and bounded below by zero). Note that, in comparison to previous existence arguments, we do not realize the F -signature as a relative Hilbert-Kunz multiplicity (*cf.* Remark 4.18); rather, it is a limit of (appropriately scaled) Hilbert-Kunz multiplicities of a sequence of naturally defined ideals.

In his work [Yao06], Y. Yao has generalized the F -signature to arbitrary local rings (R, \mathfrak{m}, k) in prime characteristic $p > 0$ without the simplifying assumptions that R is complete and the residue field k is perfect. The existence of the F -signature in full generality, however, immediately reduces to the case of F -finite local rings originally considered in [HL02] (*cf.* Remark 4.10). As such, we have largely restricted ourselves to the F -finite setting throughout. Under this hypothesis, one incorporates $\alpha(R) = \log_p[k : k^p]$ into the definition $\lim_{e \rightarrow \infty} \frac{a_e}{p^{e(d+\alpha(R))}}$ of the F -signature when the residue field k is not perfect [AL03].

It has been noted that in almost all previously known cases of the existence of F -signature, the conjectured equivalence of strong and weak F -regularity holds. This observation has lead to much speculation concerning a connection between this conjecture and the existence

of F -signature; however, we are as of yet unaware of an application of our results or methods in this direction (*cf.* Remark 4.18).

Much of the study of F -signature to date has focused simply on the existence of this invariant. With this chapter closed, however, we would argue that the subject is now ripe for new exploration. It is our hope that this is but another beginning in the use of F -signature to better understand local rings in positive characteristic.

The author would like to thank Manuel Blickle and Karl Schwede for discussions and encouragement related to this article, as well as Craig Huneke for sharing an insightful observation (*cf.* Remark 3.8) after seeing a preliminary draft.

2. BACKGROUND AND NOTATION

Throughout this paper, we shall assume all rings are commutative with a unit, Noetherian, and have prime characteristic $p > 0$. A local ring is a triple (R, \mathfrak{m}, k) where \mathfrak{m} is the unique maximal ideal of the ring R and $k = R/\mathfrak{m}$ is the corresponding residue field. The Frobenius or p -th power endomorphism $F: R \rightarrow R$ is defined by $r \mapsto r^p$ for all $r \in R$. Similarly, for $e \in \mathbb{N}$, we have $F^e: R \rightarrow R$ given by $r \mapsto r^{p^e}$.

Let M be an R -module. For any $e \in \mathbb{N}$, viewing M as an R -module via restriction of scalars under F^e yields an R -module we denote by $F_*^e M$. Thus, $F_*^e M$ agrees with M as an abelian group, and if $m \in M$ we set $F_*^e m$ to be the corresponding element of $F_*^e M$. Furthermore, for $r \in R$ it follows that $r(F_*^e m) = F_*^e(r^{p^e} m)$. Note that $F_*^e R$ inherits the structure of a ring abstractly isomorphic to R , and $F_*^e M$ is naturally an $F_*^e R$ -module for any R -module M .

We have that $F_*^e R$ is an R -algebra via the homomorphism of R -modules $F^e: R \rightarrow F_*^e R$ given by $r \mapsto F_*^e r^{p^e}$ for $r \in R$, which is but another perspective on the e -th iterate of Frobenius. In case R is reduced, we may identify $F_*^e R$ with the R -module R^{1/p^e} of p^e -th roots of R by associating $F_*^e r$ and r^{1/p^e} ; the e -iterated Frobenius homomorphism now takes on the guise of the natural inclusion $R \subseteq R^{1/p^e}$. Each point of view has certain advantages, and we will switch between them as the situation warrants throughout.

Definition 2.1. Suppose (R, \mathfrak{m}, k) is a local ring of characteristic $p > 0$. We say R is *F-finite* if $F_* R$ is finitely generated as an R -module, from which it follows that $F_*^e R$ is finitely generated for all $e \in \mathbb{N}$. In this case, we set $\alpha(R) = \log_p[k : k^p]$.

Note that any local ring which is essentially of finite type over a perfect field is F -finite. Additionally, as used for simplicity in the introduction, a complete local ring with F -finite residue field is automatically F -finite.

Denote by $\ell_R(M)$ the length of a finitely generated Artinian R -module M . If R is F -finite and $e \in \mathbb{N}$, it is easy to see that

$$\ell_R(M) = \ell_{F_*^e R}(F_*^e M) \quad \ell_R(F_*^e M) = p^{e\alpha(R)} \ell_R(M)$$

by using that $F_*^e(_)$ is an exact functor and $[(F_*^e k \simeq k^{1/p^e}) : k] = p^{e\alpha(R)}$. The following Theorems of Kunz shall be used repeatedly. A secondary reference for these Theorems can be found in the appendix to [Mat80].

Theorem 2.2. (*Kunz's Theorems*) *Let (R, \mathfrak{m}, k) be a Noetherian local ring of dimension d and characteristic $p > 0$.*

- (i) [Kun69] *For any $e \in \mathbb{N}$, we have $\ell(R/\mathfrak{m}^{[p]}) \geq p^{ed}$ and equality holds if and only if R is regular. If R is F -finite, then R^{1/p^e} is a free module of rank $p^{e(d+\alpha(R))}$ if and only if R is regular.*
- (ii) [Kun76]³ *If R is F -finite, then R is excellent and $\alpha(R_P) = \alpha(R_Q) + \dim(R_Q/PR_Q)$ for any two prime ideals $P \subseteq Q$ of R .*

We close this section by recalling the definitions of F -purity and strong F -regularity for F -finite local rings. Though first introduced in [HR76] and [HH89], respectively, both concepts have been studied by many authors. As little of the theory will be used in subsequent sections, we content ourselves to recall a few standard facts without proofs or references. The interested reader is invited to see [Hun96] or [Hoc07] for further information.

Definition 2.3. Suppose (R, \mathfrak{m}, k) is an F -finite local ring with prime characteristic $p > 0$.

- We say R is F -pure or⁴ F -split if the Frobenius homomorphism $F: R \rightarrow F_*R$ splits as a map of R -modules. In other words, there exists $\phi \in \text{Hom}_R(F_*R, R)$ such that $\phi \circ F = \text{Id}_R \in \text{Hom}_R(R, R)$. In case R is F -pure, it is automatically reduced and weakly normal, and the \mathfrak{m} -adic completion \hat{R} is also F -pure.
- Let R^0 be the complement of the minimal primes of R . We say R is *strongly F -regular* if for every $c \in R^0$ there exists an $e \geq 0$ and some $\phi \in \text{Hom}_R(F_*^e R, R)$ such that $\phi(F_*^e c) = 1$. In other words the R -linear map $R \rightarrow F_*^e R$ which sends 1 to $F_*^e c$ splits over R . In case R is strongly F -regular, it is a Cohen-Macaulay normal domain, and the \mathfrak{m} -adic completion \hat{R} is also strongly F -regular.

Remark 2.4. The notions of F -purity and F -regularity play a prominent role in the celebrated theory of tight closure introduced by M. Hochster and C. Huneke; see [HH90] for a first glance at this beautiful subject. They have conjectured that all ideals of R are tightly closed if and only if R is strongly F -regular. This is known to be true when R is an excellent \mathbb{Q} -Gorenstein normal local ring ([AM99]; cf. [LS99, LS01] and [HY03, Theorem 1.13]).

3. UNIFORM HILBERT-KUNZ ESTIMATES

Our goal in this section is to revisit the proof of a famous result of P. Monsky. For an ideal I of a local ring (R, \mathfrak{m}, k) and $e \in \mathbb{N}$, recall that $I^{[p^e]}$ denotes the ideal $\langle x^{p^e} \mid x \in I \rangle$.

³We caution the reader that Kunz states a nearby and related result in more generality than his proof justifies. See [EY10, page 4] for further details (cf. [SB79]).

⁴While technically speaking F -purity is an a priori weaker condition than F -splitting, the two notions coincide for F -finite rings; see [HR76].

Furthermore, we have

$$R/I \otimes_R F_*^e R = F_*^e R/I F_*^e R = F_*^e R/F_*^e I^{[p^e]} = F_*^e (R/I^{[p^e]})$$

by the definition of the action of R on $F_*^e R$.

Theorem 3.1. [Mon83] *Suppose (R, \mathfrak{m}, k) is a local ring of dimension d and characteristic $p > 0$. If I is any \mathfrak{m} -primary ideal and M is a finitely generated R -module, then the limit*

$$e_{HK}(I; M) := \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/I^{[p^e]} \otimes_R M)$$

exists and is called the Hilbert-Kunz multiplicity of M along I .

The proof of Theorem 3.1 given here is but a slight variant of Monsky's original proof. However, in the process, we will recover certain uniform approximation statements (see Theorem 3.6) which will be essential in showing the existence of the F -signature. We begin with a pair of rather elementary and well-known lemmas.

Lemma 3.2. *Suppose (R, \mathfrak{m}, k) is a local ring of characteristic $p > 0$. If M is a finitely generated R -module, then there exists a positive constant C such that for all $e \in \mathbb{N}$ and any ideal I of R with $\mathfrak{m}^{[p^e]} \subseteq I$ we have*

$$\ell_R(R/I \otimes_R M) \leq Cp^{e \dim(M)} .$$

Proof. Since $R/\mathfrak{m}^{[p^e]} \otimes_R M$ surjects onto $R/I \otimes_R M$, it suffices to show the statement for $I = \mathfrak{m}^{[p^e]}$. If \mathfrak{m} has t generators, then $\mathfrak{m}^{tp^e} \subseteq \mathfrak{m}^{[p^e]}$ and hence

$$\ell_R(R/\mathfrak{m}^{[p^e]} \otimes_R M) \leq \ell_R(R/(\mathfrak{m}^t)^{p^e} \otimes_R M) .$$

The Hilbert polynomial of M with respect to \mathfrak{m}^t has degree $\dim(M)$. If leading coefficient of this polynomial is c , it is clear that any $C \gg c$ satisfies the desired bound. \square

Lemma 3.3. *Suppose (R, \mathfrak{m}, k) is a d -dimensional reduced local ring of characteristic $p > 0$. Let P_1, \dots, P_m be those minimal primes of R with $\dim(R/P_i) = d$. If M and N are finitely generated R -modules such that $M_{P_i} \simeq N_{P_i}$ for each i , then there exists a positive constant C such that for all $e \in \mathbb{N}$ and any ideal I of R with $\mathfrak{m}^{[p^e]} \subseteq I$ we have*

$$|\ell_R(R/I \otimes_R M) - \ell_R(R/I \otimes_R N)| \leq Cp^{e(d-1)} .$$

Proof. Let $W = R \setminus (\cup_i P_i)$, so that $W^{-1}R = R_{P_1} \times \dots \times R_{P_m}$ and we have $W^{-1}M \simeq W^{-1}N$. Since $W^{-1} \operatorname{Hom}_R(M, N) = \operatorname{Hom}_{W^{-1}R}(W^{-1}M, W^{-1}N)$, there is some $\phi \in \operatorname{Hom}_R(M, N)$ such that $W^{-1}\phi$ is an isomorphism. Since $\operatorname{coker}(\phi)$ satisfies $W^{-1} \operatorname{coker}(\phi) = 0$ and thus has dimension strictly smaller than d , we can find a positive constant C such that for all $e \in \mathbb{N}$ and any ideal I of R containing $\mathfrak{m}^{[p^e]}$ we have

$$|\ell_R(R/I \otimes_R \operatorname{coker}(\phi))| \leq Cp^{e(d-1)} .$$

The sequence

$$R/I \otimes_R M \xrightarrow{\phi} R/I \otimes_R N \longrightarrow R/I \otimes_R \operatorname{coker}(\phi) \longrightarrow 0$$

is exact, so that

$$\ell_R(R/I \otimes_R N) - \ell_R(R/I \otimes_R M) \leq \ell_R(R/I \otimes_R \operatorname{coker}(\phi)) .$$

The lemma now follows by reversing the roles of M and N and applying the preceding lemma. \square

The next proposition is the key result from which both the existence of Hilbert-Kunz multiplicity and F -signature will follow. The essential point is that the constant C depends only on the R -module M in question.

Proposition 3.4. *Suppose (R, \mathfrak{m}, k) is a d -dimensional reduced F -finite local ring of characteristic $p > 0$. If M is a finitely generated R -module, then there exists a positive constant C such that for all $e, e' \in \mathbb{N}$ and any ideal I of R with $\mathfrak{m}^{[p^e]} \subseteq I$ we have*

$$(1) \quad \left| \ell_R(R/I \otimes_R M) - \frac{1}{p^{e'd}} \ell_R(R/I^{[p^{e'}]} \otimes_R M) \right| \leq Cp^{e(d-1)} .$$

Proof. If P_1, \dots, P_m are the minimal primes of R with $\dim(R/P_i) = d$ and $K_i = R_{P_i}$, then we have by Theorem 2.2 (ii) that $\alpha(R_{P_i}) = d + \alpha(R)$. We claim that $\bigoplus_{p^{d+\alpha(R)}} M$ and $F_* M$ are isomorphic after localizing at any of the P_i . Since K_i is a field, this follows from

$$\ell_{R_{P_i}}((F_* M)_{P_i}) = \ell_{R_{P_i}}(F_*(M_{P_i})) = p^{\alpha(R_{P_i})} \ell_{R_{P_i}}(M_{P_i}) .$$

Thus, by Lemma 3.3, there is a positive constant D such that for all $e \in \mathbb{N}$ and any ideal I of R containing $\mathfrak{m}^{[p^e]}$ we have

$$\left| \ell_R(R/I \otimes_R F_* M) - p^{d+\alpha(R)} \ell_R(R/I \otimes_R M) \right| \leq Dp^{e(d-1)} .$$

Since

$$\ell_R(R/I \otimes_R F_* M) = \ell_R\left(F_*\left(R/I^{[p]} \otimes_R M\right)\right) = p^{\alpha(R)} \ell_R(R/I^{[p]} \otimes_R M)$$

we have, setting $E = Dp^{-\alpha(R)}$,

$$\left| \ell_R(R/I^{[p]} \otimes_R M) - p^d \ell_R(R/I \otimes_R M) \right| \leq Ep^{e(d-1)}$$

for all ideals I of R with $\mathfrak{m}^{[p^e]} \subseteq I$. Let us now show by induction on $e' \in \mathbb{N}$ that

$$(2) \quad \left| \ell_R(R/I^{[p^{e'}]} \otimes_R M) - p^{e'd} \ell_R(R/I \otimes_R M) \right| \leq Ep^{(e+e'-1)(d-1)}(1 + p + \dots + p^{e'-1}) .$$

Indeed, we have

$$(3) \quad \left| \ell_R(R/I^{[p^{e'}]} \otimes_R M) - p^d \ell_R(R/I^{[p^{e'-1}]} \otimes_R M) \right| \leq Ep^{(e+e'-1)(d-1)}$$

since $\mathfrak{m}^{[p^{e+e'-1}]} \subseteq I^{[p^{e'-1}]}$. By the induction assumption, we have

$$\left| \ell_R(R/I^{[p^{e'-1}]} \otimes_R M) - p^{(e'-1)d} \ell_R(R/I \otimes_R M) \right| \leq Ep^{(e+e'-2)(d-1)}(1 + p + \dots + p^{e'-2})$$

and multiplying through by $p^d = p^{d-1}p$ gives

$$(4) \quad \left| p^d \ell_R(R/I^{[p^{e'-1}]} \otimes_R M) - p^{e'd} \ell_R(R/I \otimes_R M) \right| \leq E p^{(e+e'-1)(d-1)} (p + \dots + p^{e'-1}) .$$

Adding (3) and (4) together completes the induction and yields (2).

To finish the proof, dividing (2) through by $p^{e'd}$ shows that

$$\begin{aligned} \left| \ell_R(R/I \otimes_R M) - \frac{1}{p^{e'd}} \ell_R(R/I^{[p^{e'}]} \otimes_R M) \right| &\leq \frac{E \left(\sum_{i=0}^{e'-1} p^i \right) p^{(e+e'-1)(d-1)}}{p^{e'd}} \\ &= E \cdot \frac{p^{e'} - 1}{p - 1} \cdot \frac{p^{e(d-1)}}{p^{e'}} \cdot \frac{p^{(e'-1)(d-1)}}{p^{e'(d-1)}} \\ &\leq \left(\frac{E}{(p-1)p^{d-1}} \right) p^{e(d-1)} \end{aligned}$$

so that $C := \left(\frac{E}{(p-1)p^{d-1}} \right)$ is a positive constant satisfying the desired bound. \square

Corollary 3.5. *Suppose (R, \mathfrak{m}, k) is any local ring of dimension d and characteristic $p > 0$. Then there is an $e_0 \in \mathbb{Z}_{\geq 0}$ (depending only on R) with the following property: for every finitely generated R -module M , there exists a positive constant C so that for all $e, e' \in \mathbb{N}$ and any ideal I of R with $\mathfrak{m}^{[p^e]} \subseteq I$ we have*

$$(5) \quad \left| \ell(R/I^{[p^{e_0}]} \otimes_R M) - \frac{1}{p^{e'd}} \ell(R/I^{[p^{e'+e_0}]} \otimes_R M) \right| \leq C p^{e(d-1)} .$$

Furthermore, if R is reduced and F -finite, we may take $e_0 = 0$.

Proof. Let us first reduce to the case where R is F -finite. After picking a coefficient field for \hat{R} and generators x_1, \dots, x_n for \mathfrak{m} , let (S, \mathfrak{n}, l) be the complete faithfully flat⁵ local ring extension of (R, \mathfrak{m}, k) with $\mathfrak{m}S = \mathfrak{n}$ and $l = l^p$ given by

$$R \rightarrow \hat{R} \rightarrow \hat{R} \otimes_{k[[T_1, \dots, T_n]]} k^\infty[[T_1, \dots, T_n]] =: S$$

where $k[[T_1, \dots, T_n]] \rightarrow \hat{R}$ maps $T_i \mapsto x_i$ and $l = k^\infty$ is a perfect closure of k . If $S_{\text{red}} = S/\text{Nil}(S)$ where $\text{Nil}(S)$ is the nilradical of S , then S_{red} is reduced and F -finite of dimension d . Choose $e_0 \gg 0$ so that $(\text{Nil}(S))^{[p^{e_0}]} = 0$.

If $\mathfrak{m}^{[p^e]} \subseteq I \subseteq R$, then we have $(\mathfrak{n}S_{\text{red}})^{[p^e]} \subseteq IS_{\text{red}}$. By Proposition 3.4, there is a positive constant C so that

$$\left| \ell_{S_{\text{red}}}(S_{\text{red}}/IS_{\text{red}} \otimes_R F_*^{e_0} M) - \frac{1}{p^{e'd}} \ell_{S_{\text{red}}}(S_{\text{red}}/I^{[p^{e'}]} S_{\text{red}} \otimes_R F_*^{e_0} M) \right| \leq C p^{e(d-1)} .$$

Since we have

$$\ell_R(R/I^{[p^{e_0}]} \otimes_R M) = \ell_{S_{\text{red}}}(S_{\text{red}}/IS_{\text{red}} \otimes_R F_*^{e_0} M)$$

⁵The ring extension $k[[T_1, \dots, T_n]] \subseteq k^\infty[[T_1, \dots, T_n]]$ is flat. Since, by [Bou98, Chapter III §5] (cf. [Mat80, Theorem 49]), the flatness of a local homomorphism of local rings is preserved after completion, we see that $k[[T_1, \dots, T_n]] \subseteq k^\infty[[T_1, \dots, T_n]]$ is flat as well. Thus, the given extension $R \rightarrow S$ is faithfully flat since it is local, completion is flat, and flatness is stable under arbitrary base change.

$$\ell_R(R/I^{[p^{e'}+e_0]} \otimes_R M) = \ell_{S_{\text{red}}}(S_{\text{red}}/I^{[p^{e'}]} S_{\text{red}} \otimes_R F_*^{e_0} M)$$

the desired result now follows. \square

Proof of Theorem 3.1. Fix e_0 and C as in Corollary 3.5, and $\epsilon > 0$. Find $e_1 \gg 0$ so that $\mathfrak{m}^{[p^{e_1}]} \subseteq I$. Choose $E \gg 0$ so that $\frac{Cp^{e_1}}{p^{e_0d}p^E} < \epsilon$. If $e \geq E$, then the estimate in (5) for the ideal $I^{[p^e]} \supseteq \mathfrak{m}^{[p^{e_1+e}]}$ and any $e' \in \mathbb{N}$ after dividing by $p^{(e+e_0)d}$ yields

$$\begin{aligned} & \left| \frac{1}{p^{(e+e_0)d}} \ell_R(R/I^{[p^{e+e_0}]} \otimes_R M) - \frac{1}{p^{(e'+e+e_0)d}} \ell_R(R/I^{[p^{e'+e+e_0}]} \otimes_R M) \right| \\ & \leq \frac{Cp^{(e+e_1)(d-1)}}{p^{(e+e_0)d}} = \frac{Cp^{e_1(d-1)}}{p^{e_0d}p^e} \leq \frac{Cp^{e_1(d-1)}}{p^{e_0d}p^E} < \epsilon \end{aligned}$$

In particular, this shows the sequence $\{\frac{1}{p^{ed}} \ell_R(R/I^{[p^e]} \otimes_R M)\}_{e \in \mathbb{N}}$ is Cauchy. \square

We view the next theorem as a kind of uniform approximation statement for Hilbert-Kunz multiplicities. The subsequent corollary is the precise statement which will be needed to show the existence of the F -signature.

Theorem 3.6. *Suppose (R, \mathfrak{m}, k) is any local ring of dimension d and characteristic $p > 0$. Then there is an $e_0 \in \mathbb{Z}_{\geq 0}$ (depending only on R) with the following property: for every finitely generated R -module M , there exists a positive constant C so that for all $e \in \mathbb{N}$ and any ideal I of R with $\mathfrak{m}^{[p^e]} \subseteq I$ we have*

$$\left| \frac{1}{p^{e_0d}} \ell_R(R/I^{[p^{e_0}]} \otimes_R M) - e_{HK}(I; M) \right| \leq Cp^{e(d-1)}$$

Furthermore, if R is reduced and F -finite, we may take $e_0 = 0$.

Proof. Letting $e' \rightarrow \infty$ in Corollary 3.5 gives

$$\left| \ell(R/I^{[p^{e_0}]} \otimes_R M) - e_{HK}(I^{[p^{e_0}]}; M) \right| \leq Cp^{e(d-1)}.$$

After dividing through by p^{e_0d} , the desired result now follows immediately from

$$e_{HK}(I^{[p^{e_0}]}; M) = p^{e_0d} e_{HK}(I; M)$$

after replacing C with $\frac{1}{p^{e_0d}}C$. \square

Corollary 3.7. *Suppose (R, \mathfrak{m}, k) is a d -dimensional F -finite reduced local ring of characteristic $p > 0$, M is a finitely generated R -module, and $\{I_e\}_{e \in \mathbb{N}}$ is any sequence of ideals such that $\mathfrak{m}^{[p^e]} \subseteq I_e$. Then*

$$\lim_{e \rightarrow \infty} \frac{1}{p^{ed}} (\ell_R(R/I_e \otimes_R M) - e_{HK}(I_e; M)) = 0.$$

Remark 3.8. The uniform Hilbert-Kunz estimates of this section can also be shown using ideas from [Dut83]. More precisely, suppose R is an F -finite local domain of characteristic $p > 0$ with dimension d having perfect residue field k . Then there exists a positive constant C and a finite set Λ of nonzero prime ideals of R with the following property:

For all $e \in \mathbb{N}$ there are free R -modules F_e and G_e of rank p^{ed} together with inclusions

$$F_e \subseteq R^{1/p^e} \subseteq G_e$$

such that each of the quotients $G_e/R^{1/p^e}$ and $R^{1/p^e}/F_e$ has a prime cyclic filtration by at most Cp^{ed} copies of the various R/Q for $Q \in \Lambda$.

A proof follows in the same manner as the solution to [Hun96, Exercise 10.4], included in the appendix and due to Karen Smith. The author is grateful to Craig Huneke for pointing out the relevance of this exercise after viewing a preliminary version of this article.

4. F -SIGNATURE

4.1. Terminology and key lemmas.

Definition 4.1. Let (R, \mathfrak{m}, k) be a reduced F -finite local ring of prime characteristic $p > 0$. For each $e \in \mathbb{N}$, the e -th Frobenius splitting number of R is the largest rank $a_e = a_e(R)$ of a free R -module appearing in a direct sum decomposition of $F_*^e R$. In other words, we may write $F_*^e R = R^{\oplus a_e} \oplus M_e$ where M_e has no free direct summands.

Remark 4.2. We have that R is F -pure if and only if $a_e > 0$ for some $e \in \mathbb{N}$, in which case $a_e > 0$ for all $e \in \mathbb{N}$. If \widehat{R} is the \mathfrak{m} -adic completion of R , then it follows from $F_*^e \widehat{R} = \widehat{R} \otimes_R F_*^e R$ that the e -th Frobenius splitting numbers of R and \widehat{R} coincide. Since \widehat{R} satisfies the Krull-Schmidt condition [Swa68, Theorem 2.22], a direct sum decomposition of $F_*^e \widehat{R}$ as in the above definition is unique up to isomorphism. As a result, the values a_e for any F -finite local ring are independent of the given direct sum decomposition above. Alternatively, Proposition 4.5 below can be seen as an elementary proof of this assertion.

As indicated in the introduction, we aim to show that the sequence $\{\frac{a_e}{p^{e(d+\alpha(R))}}\}_{e \in \mathbb{N}}$ approaches a limit. This will be done by applying the uniform Hilbert-Kunz estimates from the previous section to the following collection of naturally defined ideals.

Definition 4.3. Suppose (R, \mathfrak{m}, k) is an F -finite local ring of prime characteristic $p > 0$. For each $e \in \mathbb{N}$, we define $I_e = \{r \in R \mid \phi(F_*^e r) \in \mathfrak{m} \text{ for all } \phi \in \text{Hom}_R(F_*^e R, R)\}$.

Lemma 4.4. Suppose (R, \mathfrak{m}, k) is a reduced F -finite local ring with characteristic $p > 0$. Then I_e is an ideal containing $\mathfrak{m}^{[p^e]}$. Furthermore, we have $I_e^{[p]} \subseteq I_{e+1}$.

Proof. We leave it to the reader to verify that I_e is an ideal of R , and also $\mathfrak{m}^{[p^e]} \subseteq I_e$ for all $e \in \mathbb{N}$. Suppose now $\phi \in \text{Hom}_R(R^{1/p^{e+1}}, R)$ and $r \in I_e$. Then it follows

$$\phi((r^p)^{1/p^{e+1}}) = (\phi|_{R^{1/p^e}})(r^{1/p^e}) \in \mathfrak{m}$$

as $\phi|_{R^{1/p^e}} \in \text{Hom}_R(R^{1/p^e}, R)$ and $r \in I_e$. Thus, we see $r^p \in I_{e+1}$ as desired. \square

Proposition 4.5. [AE05, Corollary 2.8][Yao06, Lemma 2.1] Suppose (R, \mathfrak{m}, k) is an F -finite local ring with prime characteristic $p > 0$. If $F_*^e R = R^{\oplus a_e} \oplus M_e$ is a direct sum decomposition

where M_e has no free direct summands, then

$$a_e = \ell_R(F_*^e(R/I_e)) = \ell_R(\operatorname{Hom}_R(F_*^e R, R) / \operatorname{Hom}_R(F_*^e R, \mathfrak{m}))$$

Proof. The assumption that M_e has no free direct summands implies that $\phi(M_e) \subseteq \mathfrak{m}$ for all $\phi \in \operatorname{Hom}_R(M_e, R)$, or equivalently all $\phi \in \operatorname{Hom}_R(F_*^e R, R) = \operatorname{Hom}_R(R^{\oplus a_e}, R) \oplus \operatorname{Hom}_R(M_e, R)$. It is easy to see from the definition of I_e that

$$F_*^e I_e = \mathfrak{m}^{\oplus a_e} \oplus M_e$$

and thus $F_*^e(R/I_e) = F_*^e R / F_*^e I_e \simeq k^{\oplus a_e}$ has length a_e . Similarly, we have

$$\begin{aligned} \operatorname{Hom}_R(F_*^e R, \mathfrak{m}) &= \{\phi \in \operatorname{Hom}_R(F_*^e R, R) \mid \phi(F_*^e R) \subseteq \mathfrak{m}\} \\ &= (\mathfrak{m} \operatorname{Hom}_R(R^{\oplus a_e}, R)) \oplus \operatorname{Hom}_R(M_e, R) \end{aligned}$$

so that also $\operatorname{Hom}_R(F_*^e R, R) / \operatorname{Hom}_R(F_*^e R, \mathfrak{m}) \simeq k \otimes_R \operatorname{Hom}_R(R^{\oplus a_e}, R)$ has length a_e . \square

Remark 4.6. The ideals I_e appear in the works of Y. Yao [Yao06] as well as those of I. Aberbach and F. Enescu [AE05], albeit with a different formulation. For completeness, let us recover their description, which will not be needed in the remainder of this article. We assume (R, \mathfrak{m}, k) is F -finite and complete. Let $E = E_R(k)$ denote the injective hull of the residue field, and $(_)^\vee = \operatorname{Hom}_R(_, E)$ the Matlis duality functor. If $u \in E$ is a generator for the socle, we have $k = Ru \subseteq E$. There are isomorphisms $E^\vee \simeq R$ and $(E/k)^\vee \simeq \mathfrak{m}$, whereby the natural map $(E/k)^\vee \rightarrow E^\vee$ corresponds to the inclusion $\mathfrak{m} \subseteq R$. Thus, we have a commutative diagram of $F_*^e R$ -modules

$$\begin{array}{ccccc} \operatorname{Hom}_R(F_*^e R, \mathfrak{m}) & \longrightarrow & \operatorname{Hom}_R(F_*^e R, (E/k)^\vee) & \longrightarrow & (F_*^e R \otimes_R (E/k))^\vee \\ \phi \downarrow & & \downarrow & & \downarrow \psi \\ \operatorname{Hom}_R(F_*^e R, R) & \longrightarrow & \operatorname{Hom}_R(F_*^e R, E^\vee) & \longrightarrow & (F_*^e R \otimes_R E)^\vee \end{array}$$

where each of the vertical arrows is an inclusion, and the horizontal arrows are isomorphisms. By definition, $F_*^e I_e = \operatorname{Ann}_{F_*^e R}(\operatorname{coker}(\phi)) = \operatorname{Ann}_{F_*^e R}(\operatorname{coker}(\psi))$, so it follows that $F_*^e I_e = \operatorname{Ann}_{F_*^e R}(\ker(\psi^\vee))$. Since we have an exact sequence

$$F_*^e R \otimes_R k \longrightarrow F_*^e R \otimes_R E \xrightarrow{\psi^\vee} F_*^e R \otimes_R (E/k) \longrightarrow 0$$

we recover the description

$$I_e = \{r \in R \mid F_*^e r \otimes u = 0 \text{ in } F_*^e R \otimes_R E\}.$$

The following lemma was first observed by I. Aberbach and F. Enescu, and a simplified proof has been included for completeness.

Lemma 4.7. [AE05, Theorem 1.1] [Sch08, Remark 4.4] *Suppose the local ring (R, \mathfrak{m}, k) is reduced and F -finite. The ideal*

$$P = \bigcap_{e \in \mathbb{N}} I_e$$

is either prime or all of R , and is called the *splitting prime* of R .

Proof. Supposing $c_1, c_2 \in R \setminus P$, we can find $\phi_i \in \text{Hom}_R(R^{1/p^{e_i}}, R)$ for some $e_1, e_2 \in \mathbb{N}$ with $\phi_i((c_i)^{1/p^{e_i}}) = 1$ for $i = 1, 2$. But then

$$\phi := \phi_1 \circ \phi_2^{1/p^{e_1}} \left(c_1^{1/p^{e_1} - 1/p^{e_1+e_2}} \cdot (_) \right) \in \text{Hom}_R(R^{1/p^{e_1+e_2}}, R)$$

satisfies $\phi((c_1 c_2)^{1/p^{e_1+e_2}}) = 1$ so that $c_1 c_2 \in R \setminus P$. \square

Remark 4.8. It is immediate that $P \neq R$ precisely when R is F -pure, and $P = 0$ if and only if R is strongly F -regular. Furthermore, it is straightforward to show R/P is strongly F -regular [AE05, Theorem 4.7][Sch08, Corollary 7.8]. The prime P is called the *splitting prime* of R , and can also be viewed as the unique largest center of F -purity in R ; see [Sch08, Remark 4.4]. The F -*splitting ratio* appearing below was first introduced in [AE05], though the equivalence of the original definition with that which follows makes use of Theorem 4.26.

4.2. Theorem statements and proofs.

Theorem 4.9. *Let (R, \mathfrak{m}, k) be a d -dimensional F -finite local ring with prime characteristic $p > 0$ and $\alpha(R) = [k : k^p]$. Then the limit*

$$s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e(d+\alpha(R))}}$$

exists and is called the F -signature of R . More generally, if P is the splitting prime of R , then the limit

$$r_F(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e(\dim(R/P) + \alpha(R))}}$$

exists and is called the F -splitting ratio of R .

Proof. If R is not F -pure, then $a_e = 0$ for all $e \in \mathbb{N}$ and both statements are clear. Thus, we may assume R is F -pure and hence reduced.

Let us first show the existence of F -signature using only the two properties of the ideals I_e shown in Lemma 4.4. Since $\mathfrak{m}^{[p^e]} \subseteq I_e$, it follows from Corollary 3.7 that

$$\lim_{e \rightarrow \infty} \left(\frac{1}{p^{ed}} (\ell(R/I_e) - e_{HK}(I_e; R)) \right) = 0.$$

From Proposition 4.5, we see that

$$\frac{a_e}{p^{e(d+\alpha(R))}} = \frac{1}{p^{ed}} \ell_R(R/I_e)$$

for all $e \in \mathbb{N}$. Thus, to prove the existence of the F -signature it suffices to show the sequence $\{\frac{1}{p^{ed}} e_{HK}(I_e; R)\}_{e \in \mathbb{N}}$ approaches a limit. Since $I_e^{[p]} \subseteq I_{e+1}$, we have

$$e_{HK}(I_{e+1}; R) \leq e_{HK}(I_e^{[p]}; R) = p^d (e_{HK}(I_e; R))$$

and dividing through by $p^{(e+1)d}$ gives

$$\frac{1}{p^{(e+1)d}} e_{HK}(I_{e+1}; R) \leq \frac{1}{p^{ed}} e_{HK}(I_e; R)$$

for all $e \in \mathbb{N}$. Since $\{\frac{1}{p^{ed}}e_{HK}(I_e; R)\}_{e \in \mathbb{N}}$ is non-increasing (and bounded below by zero), the desired conclusion follows at once.

More generally for the F -splitting ratio, let $\overline{R} = R/P$ and for any ideal $I \subseteq R$ set $\overline{I} = I \cdot \overline{R}$. Since $\mathfrak{m}^{[p^e]} \subseteq I_e$ and $I_e^{[p]} \subseteq I_{e+1}$, it follows immediately that $\overline{\mathfrak{m}}^{[p^e]} \subseteq \overline{I}_e$ and $\overline{I}_e^{[p]} \subseteq \overline{I}_{e+1}$. Thus, the preceding argument applied to the ideals \overline{I}_e shows the existence of the limit

$$\lim_{e \rightarrow \infty} \frac{\ell_{\overline{R}}(\overline{R}/\overline{I}_e)}{p^{e(\dim(R/P))}} = \lim_{e \rightarrow \infty} \frac{e_{HK}(\overline{I}_e; \overline{R})}{p^{e(\dim(R/P))}}.$$

Since $a_e = p^{e(\alpha(R))} \ell_R(R/I_e) = p^{e(\alpha(R))} \ell_{\overline{R}}(\overline{R}/\overline{I}_e)$, this limit is precisely the F -splitting ratio of R . \square

Remark 4.10. When (R, \mathfrak{m}, k) is not necessarily F -finite, Y. Yao [Yao06, Remark 2.3] uses the description from 4.6 to define the F -signature. However, the existence of the F -signature in this setting immediately reduces to the F -finite case shown in Theorem 4.9.

Theorem 4.11. *Let (R, \mathfrak{m}, k) be a d -dimensional F -finite characteristic $p > 0$ local domain and let M be a finitely generated R -module. Denote by b_e the maximal rank of a free R -module appearing in a direct sum decomposition of $F_*^e M$. Then*

$$\lim_{e \rightarrow \infty} \frac{b_e}{p^{e(d+\alpha(R))}} = \text{rank}(M) \cdot s(R).$$

Proof. Set $I_e^M = \{m \in M \mid \phi(F_*^e m) \in \mathfrak{m} \text{ for all } \phi \in \text{Hom}_R(F_*^e M, R)\}$. Following the line of argument in Proposition 4.5, it is easy to see that I_e^M is an R -submodule of M with

$$b_e = \ell_R(F_*^e(M/I_e^M)) = p^{e\alpha(R)} \ell_R(M/I_e^M).$$

Furthermore, suppose $m \in M$ and $\phi \in \text{Hom}_R(F_*^e M, R)$. Since $\phi(_ \cdot F_*^e m) \in \text{Hom}_R(F_*^e R, R)$, we have that $\phi(F_*^e(rm)) \in \mathfrak{m}$ for all $r \in I_e$. Thus, $rm \in I_e^M$, and hence $I_e M \subseteq I_e^M$.

Let $G \subseteq M$ be a full rank free R -submodule, so that there exists $0 \neq c \in \text{Ann}_R(M/G)$. We will show $I_e^M \subseteq (I_e M :_M c)$. As $F_*^e G \simeq (F_*^e R)^{\oplus \text{rank}(M)}$, it is easy to see

$$\{g \in G \mid \phi(F_*^e g) \in \mathfrak{m} \text{ for all } \phi \in \text{Hom}_R(F_*^e G, R)\} = I_e G \simeq I_e^{\oplus \text{rank}(M)}.$$

Now, suppose we have $\phi \in \text{Hom}_R(F_*^e G, R)$ and $m \in I_e^M$. It follows that $\phi(F_*^e(cm)) \in \mathfrak{m}$ as $\phi(F_*^e c \cdot _) \in \text{Hom}_R(F_*^e M, R)$, whence $cM \in I_e G \subseteq I_e M$.

Using the four term exact sequence

$$0 \longrightarrow (I_e M :_M c)/I_e M \longrightarrow M/I_e M \xrightarrow{\cdot c} M/I_e M \longrightarrow M/(I_e M + cM) \longrightarrow 0$$

we have that

$$\ell_R(I_e^M/I_e M) \leq \ell_R((I_e M :_M c)/I_e M) = \ell_R(M/(I_e M + cM)).$$

Applying Lemma 3.2 now gives

$$\lim_{e \rightarrow \infty} \frac{1}{p^{ed}} (\ell_R(M/I_e M) - \ell_R(M/I_e^M)) = 0.$$

Furthermore, by Corollary 3.7 we have

$$\lim_{e \rightarrow \infty} \frac{1}{p^{ed}} (\ell_R(M/I_e M) - e_{HK}(I_e; M)) = 0 .$$

Putting everything together, we now have

$$\begin{aligned} \lim_{e \rightarrow \infty} \frac{b_e}{p^{e(d+\alpha(R))}} &= \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_R(M/I_e^M) \\ &= \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_R(M/I_e M) \\ &= \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} e_{HK}(I_e; M) \\ &= \text{rank}(M) \cdot \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} e_{HK}(I_e; R) \\ &= \text{rank}(M) \cdot s(R) \end{aligned}$$

as desired. \square

Remark 4.12. In the notation of the previous result, Y. Yao has proposed $\lim_{e \rightarrow \infty} \frac{b_e}{p^{e(d+\alpha(R))}}$ as a potential definition of F -signature for a finitely generated R -module M ; see [Yao06].

The following Corollary extends [HL02, Proposition 19] by removing the Gorenstein hypothesis. In particular, this result recovers explicit formulae (first shown in [Yao06, Remark 4.7]) for the F -signatures of arbitrary finite quotient singularities.

Corollary 4.13. *Let $(R, \mathfrak{m}, k) \subseteq (S, \mathfrak{n}, l)$ be a local inclusion of d -dimensional F -finite characteristic $p > 0$ local domains with corresponding extension of fraction fields $K = \text{Frac}(R) \subseteq \text{Frac}(S) = L$. Assume S is a finitely generated R -module and write $S = R^{\oplus f} \oplus M$ where M has no nonzero free direct summands. Then*

$$f \cdot s(S) \leq [L : K] \cdot s(R) .$$

If additionally S is regular, then equality holds and

$$s(R) = \frac{f}{[L : K]}$$

Proof. Note first that $\alpha(R) = \alpha(S)$ since $[l : l^p][l^p : k^p] = [l : k^p] = [l : k][k : k^p]$. As above, let b_e denote the maximal rank of a free R -module appearing in a direct sum decomposition of $F_*^e S$. If we write $F_*^e S = S^{\oplus a_e(S)} \oplus N_e$ as S -modules where N_e has no free direct summands, we automatically get a direct sum decomposition of $F_*^e S$

$$F_*^e S = (R^{\oplus f} \oplus M)^{\oplus a_e(S)} \oplus N_e$$

as an R -module with a free direct summand of rank $f \cdot a_e(S)$. Thus, we have $f \cdot a_e(S) \leq b_e$. Furthermore, if S is regular, then $N_e = 0$ and equality holds. Both statements now follow at once by dividing through by $p^{e(d+\alpha(R))} = p^{e(d+\alpha(S))}$ and letting $e \rightarrow \infty$. \square

We will need the following results from [HL02]; the proof given herein is due to the original authors.

Theorem 4.14. [HL02, Propositions 14 and 15] *Suppose (R, \mathfrak{m}, k) is an F -finite local ring with characteristic $p > 0$. If $I \subsetneq J$ are two \mathfrak{m} -primary ideals, then*

$$(6) \quad \frac{e_{HK}(I; R) - e_{HK}(J; R)}{\ell_R(J/I)} \geq s(R) .$$

In particular, if R is Cohen-Macaulay, it follows that

$$(e(\mathfrak{m}; R) - 1)(1 - s(R)) \geq e_{HK}(\mathfrak{m}; R) - 1$$

where $e(\mathfrak{m}; R)$ is the Hilbert-Samuel multiplicity of R along \mathfrak{m} .

Proof. If we write $F_*^e R = R^{\oplus a_e} \oplus M_e$ where M_e has no direct summands, it follows that

$$\ell_R(R/I \otimes_R F_*^e R) - \ell_R(R/J \otimes_R F_*^e R) = a_e \ell_R(J/I) + \ell_R(JM_e/IM_e) \geq a_e \ell_R(J/I) .$$

Dividing through by $\ell_R(J/I)p^{e(d+\alpha(R))}$ where $d = \dim(R)$ and letting $e \rightarrow \infty$ gives (6). For the last statement, we may assume additionally that R is complete with infinite residue field; see [Yao06, Remark 2.3(3)]. Take $J = \mathfrak{m}$ and let I be a minimal reduction of \mathfrak{m} . Since $e(\mathfrak{m}; R) = e_{HK}(I; R) = \ell_R(R/I)$, the desired result follows from (6). \square

The following result combines [HL02, Proposition 14] (Theorem 4.14 above) and Corollary 4.13 to extend [HL02, Proposition 19] by once again removing the Gorenstein hypothesis.

Corollary 4.15. *Let S be an F -finite regular local ring of characteristic p and let G be a finite group acting on S with $p \nmid |G|$. Set $R = S^G$ and write $S = R^{\oplus f} \oplus M$ as an R -module where M has no free direct summands. If R is not regular, then*

$$|G| \geq \frac{f(e(R) - 1)}{e(R) - e_{HK}(R)} .$$

4.3. Open questions and further remarks. The following Theorem summarizes some further known properties of the F -signature. Related open questions will follow immediately thereafter.

Theorem 4.16. *Suppose (R, \mathfrak{m}, k) is a d -dimensional F -finite local ring with prime characteristic $p > 0$.*

- (i) [HL02, Corollary 16] *We have $s(R) \leq 1$ with equality if and only if R is regular*
- (ii) [Yao06, Theorem 3.1] *If $d \geq 2$, then $s(R) \geq 1 - \frac{1}{d!p^d}$ if and only if R is regular.*
- (iii) [AL03, Theorem 0.2] *We have $s(R) > 0$ if and only if R is strongly F -regular.*

Proof. We show only (i) and the forward direction of (iii), referring the reader to the references above for the remainder.

Assume first that R is not strongly F -regular. Then there exists $c \in R$ not contained in any minimal prime such that $c \in I_e$ for all $e \in \mathbb{N}$. Since $\mathfrak{m}^{[p^e]} + \langle c \rangle \subseteq I_e$, we have

$$s(R) = \lim_{e \rightarrow \infty} \frac{a_e}{p^{e(d+\alpha(R))}} = \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/I_e) \leq \lim_{e \rightarrow \infty} \frac{1}{p^{ed}} \ell_R(R/\mathfrak{m}^{[p^e]} \otimes_R R/\langle c \rangle) = 0$$

since $\dim(R/\langle c \rangle) < d$.

Thus, we may assume R is strongly F -regular and hence a domain. Let $K = \text{Frac}(R)$ and for $e \in \mathbb{N}$ write $R^{1/p^e} = R^{\oplus a_e} \oplus M_e$ where M_e has no free direct summands. Then it follows $K^{1/p^e} = K \otimes_R R^{1/p^e} = K^{\oplus a_e} \oplus (K \otimes_R M_e)$. Since $[K^{1/p^e} : K] = p^{e(d+\alpha(R))}$, we must have $a_e \leq p^{e(d+\alpha(R))}$ and the inequality $s(R) \leq 1$ follows at once. In case R is regular we have $s(R) = 1$ as $R^{1/p^e} = R^{\oplus p^{e(d+\alpha(R))}}$ so that $a_e = p^{e(d+\alpha(R))}$ for all $e \in \mathbb{N}$. Conversely, if $s(R) = 1$ it follows from Theorem 4.14 that $e_{HK}(R) = 1$. Thus, by [WY00], R is regular. \square

We now conclude by remarking on some of the remaining open questions concerning the F -signature, as well as some forthcoming results on the F -splitting ratio. The first question, originally posed by K.-i. Watanabe and K.-i. Yoshida, is closely related to the conjectured equivalence of strong and weak F -regularity.

Question 4.17. [WY04, Question 1.10] If the local ring (R, \mathfrak{m}, k) is reduced and F -finite with characteristic $p > 0$, can one always find \mathfrak{m} -primary ideals $I \subsetneq J$ such that equality holds in (6)?

Remark 4.18. It has been shown by [Yao06] that the F -signature agrees with the notion of *minimal relative Hilbert-Kunz multiplicity* as defined by [WY04].

The following question is motivated by Theorem 4.16 (ii) and the analogous open question for Hilbert-Kunz multiplicities; see [BE04] and [WY05] for further details.

Question 4.19. What is

$$\sigma(p, d) = \sup\{s(R) \mid R \text{ is a } d\text{-dimensional characteristic } p \text{ non-regular local ring}\}$$

for fixed $p > 0$ and d ? For which rings R , if any, is the equality $s(R) = \sigma(p, d)$ achieved?

More generally still, one can ask:

Question 4.20. What possible values can $s(R)$ attain as R varies over all local rings of fixed characteristic $p > 0$ and dimension d ? In particular, is $s(R)$ necessarily rational?

Remark 4.21. At present, all currently known computations of F -signature have proven to be rational. These include finite quotient singularities (Theorem 4.13), affine semigroup rings [Sin05], as well as Segre products and Veronese subrings of polynomial rings [WY04]. However, a conjecture of P. Monsky would give an example of F -signature which is irrational.

Proposition 4.22. *Let $R = \mathbb{F}_2[[x, y, z, u, v]]/\langle uv + x^3 + y^3 + xyz \rangle$. Assuming [Mon08, Conjecture 1.5], it follows that $s(R) = \frac{2}{3} - \frac{5}{14\sqrt{7}}$.*

Proof. By [Mon08, Corollary 2.7], Monsky's conjecture implies $e_{HK}(\mathfrak{m}) = \frac{4}{3} + \frac{5}{14\sqrt{7}}$ where $\mathfrak{m} = \langle x, y, z, u, v \rangle$. Now, consider the parameter ideal $I = \langle x, y, z, u + v \rangle \subseteq R$. Since R is Cohen-Macaulay, it follows $e_{HK}(I) = \ell_R(R/I) = 2$. Further, the image of u generates the

socle in R/I , and we have $\langle I, u \rangle = \mathfrak{m}$. Thus, by the proof of [HL02, Theorem 11 (2)], we have

$$s(R) = e_{HK}(I) - e_{HK}(\mathfrak{m}) = 2 - \left(\frac{4}{3} + \frac{5}{14\sqrt{7}} \right) = \frac{2}{3} - \frac{5}{14\sqrt{7}}$$

as claimed. \square

Another important open question regarding F -signature asks how F -signature behaves after localization. The following observation is rather immediate: if (R, \mathfrak{m}, k) is an F -finite local ring and $\mathfrak{p} \subset \mathfrak{m}$ is a prime ideal, then $s(R_{\mathfrak{p}}) \geq s(R)$. This follows simply by taking direct sum decompositions of R^{1/p^e} and localizing at \mathfrak{p} , and one is led to the following natural question.

Question 4.23. [EY10] If (R, \mathfrak{m}, k) is an F -finite local ring, is the function

$$P \in \operatorname{Spec}(R) \mapsto s(R_P)$$

is lower semicontinuous?

Remark 4.24. F. Enescu and Y. Yao have shown that the e -th Frobenius splitting number function is lower semicontinuous [EY10].

Finally, when R is F -pure but not strongly F -regular, one may ask to what extent the F -splitting ratio $r_F(R)$ characterizes the asymptotic behavior of the sequence of F -splitting numbers. To that end, recall the following definition from [AE05].

Definition 4.25. [AE05] Let (R, \mathfrak{m}, k) be an F -finite local ring. The s -dimension or Frobenius splitting dimension of R is the largest integer k such that

$$\liminf_{e \rightarrow \infty} \frac{a_e}{p^{e(k+\alpha(R))}}$$

is not zero, where a_e is the e -th Frobenius splitting number of R .

In future work in preparation [BST11] together with M. Blickle and K. Schwede, we are able to show the following. This gives a positive answer to [AE05, Question 4.9]

Theorem 4.26. [BST11] *Suppose (R, \mathfrak{m}, k) is a reduced local F -finite ring with characteristic $p > 0$. Then the s -dimension of R is equal to $\dim(R/P)$ where P is the splitting prime of R . In other words, if R has an F -splitting, then the F -splitting ratio $r_F(R)$ is always strictly positive.*

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